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LETTER TO THE EDITOR

Canonical Bäcklund transformations and new solutions of some field equations

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Abstract. Let the Hamiltonian of the field φ be H. It is shown that the canonical transformation $g:(\varphi_1, \pi_1)\mapsto(\varphi_2, \pi_2)$ leads to a Bäcklund transformation $H(\varphi_1, \pi_1)=H^*(\varphi_2, \pi_2)$ and the latter gives some new solutions for the equations $\partial_\alpha\partial_\alpha\varphi = \mathrm{d}F_i(\varphi)/\mathrm{d}\varphi$, $i=1, 2, \ldots$.

The canonical transformations in classical mechanics have important meaning for solving the equations of motion [1]. This fact prompts us to study the problem of solving field equations using the method of canonical transformations.

In a previous paper [2], we established the general theory of canonical transformations of fields, and discussed the simple applications of the theory. In the present letter, we further study the applications of the theory, and obtain some quite interesting results. We feel that the canonical transformations are very important in field theory, as they are in mechanics.

Let T^*M be a ∞^{2n}-dimensional cotangent bundle with the coordinates $[\varphi(x), \pi(x)]$, and $R\times T^*M$ be a $(\infty^{2n}+1)$-dimensional extended phase space of fields φ with parameters $t\in\mathbb{R}$. We have defined the canonical variational 2-form

$$\tilde{\gamma}(\varphi, \pi) = \int \mathrm{d}^n x\, \tilde{\delta}\pi \wedge \tilde{\delta}\varphi \tag{1}$$

and obtained the condition of canonical transformation g: $(\varphi_1, \pi_1)\mapsto(\varphi_2, \pi_2)$ as

$$\tilde{\gamma}(\varphi_1, \pi_1) = \tilde{\gamma}(\varphi_2, \pi_2) \tag{2}$$

on the manifold T^*M [2].

On the new manifold $R\times T^*M$, we should have the extended canonical 2-form

$$\tilde{\Gamma}(\varphi, \pi) = \tilde{\gamma}(\varphi, \pi) - \tilde{\delta}H(\varphi, \pi)\wedge\tilde{\delta}t \tag{3}$$

as in the case of the differential form [3]. We can easily prove that the condition (2) of the canonical transformation becomes

$$\tilde{\Gamma}(\varphi_1, \pi_1) = \tilde{\Gamma}(\varphi_2, \pi_2) \tag{4}$$

in this case. The corresponding conditions of the 1-form may be written as

$$\tilde{\delta}G_1 = \int \mathrm{d}^n x(\pi_1\tilde{\delta}\varphi_1 - \pi_2\tilde{\delta}\varphi_2) - [H(\varphi_1, \pi_1) - H^*(\varphi_2, \pi_2)]\tilde{\delta}t \tag{5}$$

$$\tilde{\delta}G_2 = \int \mathrm{d}^n x(\varphi_1\tilde{\delta}\pi_1 - \varphi_2\tilde{\delta}\pi_2) + [H(\varphi_1, \pi_1) - H^*(\varphi_2, \pi_2)]\tilde{\delta}t \tag{6}$$

$$\tilde{\delta}G_3 = \int \mathrm{d}^n x(\pi_1\tilde{\delta}\varphi_1 + \varphi_2\tilde{\delta}\pi_2) - [H(\varphi_1, \pi_1) - H^*(\varphi_2, \pi_2)]\tilde{\delta}t \tag{7}$$

$$\tilde{\delta}G_4 = \int \mathrm{d}^n x(\varphi_1\tilde{\delta}\pi_1 + \pi_2\tilde{\delta}\varphi_2) + [H(\varphi_1, \pi_1) - H^*(\varphi_2, \pi_2)]\tilde{\delta}t \tag{8}$$

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since $\tilde{\delta}^2 G_i = 0$, where G_i $(i = 1, 2, 3, 4)$ denote the generating functional of the field's canonical transformations. From (5)-(8) we see that

$$\pi_1 = \frac{\delta G_1}{\delta \varphi_1} \qquad \pi_2 = -\frac{\delta G_1}{\delta \varphi_2} \qquad H(\varphi_1, \pi_1) - H^*(\varphi_2, \pi_2) = -\frac{\partial G_1}{\partial t} \tag{9}$$

$$\varphi_1 = \frac{\delta G_2}{\delta \pi_1} \qquad \varphi_2 = -\frac{\delta G_2}{\delta \pi_2} \qquad H(\varphi_1, \pi_1) - H^*(\varphi_2, \pi_2) = \frac{\partial G_1}{\partial t} \tag{10}$$

$$\pi_1 = \frac{\delta G_3}{\delta \varphi_1} \qquad \varphi_2 = \frac{\delta G_3}{\delta \pi_1} \qquad H(\varphi_1, \pi_1) - H^*(\varphi_2, \pi_2) = -\frac{\partial G_3}{\partial t} \tag{11}$$

$$\varphi_1 = \frac{\delta G_4}{\delta \pi_1} \qquad \pi_2 = \frac{\delta G_4}{\delta \varphi_2} \qquad H(\varphi_1, \pi_1) - H^*(\varphi_2, \pi_2) = \frac{\partial G_4}{\partial t} \tag{12}$$

which are called the equations of the field's canonical transformation.

Let us consider the case in which G_i is not an explicit function of time t. The corresponding equation of the field's canonical transformation gives

$$H(\varphi_1, \pi_1) = H^*(\varphi_2, \pi_2). \tag{13}$$

Equation (13) is defined as the canonical Bäcklund transformation (CBT) between the equations $\partial\varphi_1/\partial t = \delta H/\delta\pi_1$, $\partial\pi_1/\partial t = -\delta H/\delta\varphi_1$ and $\partial\varphi_2/\partial t = \delta H^*/\delta\pi_2$, $\partial\pi_2/\partial t = -\delta H^*/\delta\varphi_2$. By applying the CBT (13), we can obtain many interesting results.

As an example, we consider the CBT between two of the equations

$$(\partial_j\partial_j - \partial_t\partial_t)\varphi = \mathrm{d}F_i(\varphi)/\mathrm{d}\varphi \qquad i = 1, 2, \ldots \tag{14}$$

where for brevity we have used the notation $\partial_j = \partial/\partial x_j$, $j = 1, 2, n, \ldots, \partial_r = \partial_0 = \partial/\partial x_0$. Here, and throughout, we use the summation convention: a Greek index runs from 0 to n and any other index runs from 1 to $n-1$ or any other number. We know the Hamiltonian of equations (14) as

$$H = \int \mathrm{d}^n x[\tfrac{1}{2}\partial_\alpha\varphi\partial_\alpha\varphi + F_i(\varphi)]. \tag{15}$$

This and equation (13) imply the CBT of equation (14) in the form

$$\partial_\alpha\varphi_1\partial_\alpha\varphi_1 - \partial_\beta\varphi_2\partial_\beta\varphi_2 = \partial_\alpha(\varphi_1 + \varphi_2)\partial_\alpha(\varphi_1 - \varphi_2) = 2[F_i(\varphi_2) - F_j(\varphi_1)]. \tag{16}$$

If we can change the right-hand side of (16) into

$$2[F_i(\varphi_2) - F_j(\varphi_1)] = f_i(\varphi_1 + \varphi_2) f_j(\varphi_1 - \varphi_2) \tag{17}$$

then from (16) we can obtain some formulae of nonlinear superposition

$$\int \frac{\mathrm{d}(\varphi_1 + \varphi_2)}{f_i(\varphi_1 + \varphi_2)} \int \frac{\mathrm{d}(\varphi_1 - \varphi_2)}{f_j(\varphi_1 - \varphi_2)} = \int \mathrm{d}\gamma_k \int \mathrm{d}\gamma_l \qquad l = k = 1, 2, \ldots \tag{18}$$

where

$$\begin{aligned}
&\gamma_1 = A_1^{-1}(a_i x_i - D_1 t) && A_1 = (a_i a_i + D_1^2)^{1/2} \\
&\gamma_2 = (x_i x_i + t^2)^{1/2} && \\
&\gamma_3 = A_2^{-1}(\sqrt{x_i x_i} - Dt) && A_2 = (1 + D^2)^{1/2} \\
&\gamma_4 = [x_i x_i + (a x_j - bt)^2]^{1/2} && i \neq j \qquad a^2 + b^2 = 1
\end{aligned} \tag{19}$$

and so on. Some obvious examples will now be considered.

CBT *of the Klein-Gordon equation.* For this case, from (14) we have [4]

$$F(\varphi_1)=\tfrac{1}{2}\varphi_1^2 \qquad F(\varphi_2)=\tfrac{1}{2}\varphi_2^2$$
$$f(\varphi_1+\varphi_2)=\mathrm{i}(\varphi_1+\varphi_2) \qquad f(\varphi_1-\varphi_2)=\mathrm{i}(\varphi_1-\varphi_2) \tag{20}$$

in (16) and (17). Applying (20) to (18), we arrive at

$$\ln(\varphi_1+\varphi_2)\ln(\varphi_1-\varphi_2)=\ln\varphi_3\ln\varphi_4=-\gamma_k^2 \qquad k=1,2,\ldots \tag{21}$$

where $\varphi_3=\varphi_1+\varphi_2$ and $\varphi_4=\varphi_1-\varphi_2$ are also solutions of the Klein-Gordon equation, since the equation is linear. Here we have taken the integral constants as zero.

Given (21), we can, from a solution φ_3, obtain another new solution φ_4. For example, inserting the plane wave solution $\varphi_3=\exp(\pm\gamma_1)$ into (21) yields the plane-spherical solution

$$\varphi_4=\exp(\pm\gamma_k^2/\gamma_1) \qquad k=2,3,\ldots. \tag{22}$$

CBT *between the Liouville equation and the D'Alembert equation.* This case implies that

$$f_1(\varphi_1)=\mathrm{e}^{\varphi_1} \qquad F_2(\varphi_2)=0$$
$$f_1(\varphi_1+\varphi_2)=\mathrm{i}\sqrt{2}\,\mathrm{e}^{\frac{1}{2}(\varphi_1+\varphi_2)} \qquad f_1(\varphi_1-\varphi_2)=\mathrm{i}\sqrt{2}\,\mathrm{e}^{\frac{1}{2}(\varphi_1-\varphi_2)}. \tag{23}$$

Therefore equation (18) gives

$$C\,\mathrm{e}^{-\frac{1}{2}(\varphi_1+\varphi_2)}\,\mathrm{e}^{-\frac{1}{2}(\varphi_1-\varphi_2)}=C\,\mathrm{e}^{-\varphi_1}=(\gamma_k-\gamma_{k0})(\gamma'_{k0}-\gamma_k)=R_k^2 \qquad (\text{no sum on } k) \tag{24}$$

that is

$$\varphi_1=\ln C-2\ln R_k \tag{25}$$

where $R_k^2=(\gamma'_{k0}-\gamma_k)(\gamma_k-\gamma_{k0})$, C, γ_{k0} and γ'_{j0} denote the integral constants. It is interesting that the solution φ_2 of the D'Alembert equation is eliminated voluntarily and the canonical transformation between φ_1 and φ_2 is not explicit.

CBT *of the Liouville equation.* Taking $F_1(\varphi_1)=\mathrm{e}^{\varphi_1}$, $F_1(\varphi_2)=\mathrm{e}^{\varphi_2}$, this leads (16) to the CBT of the Liouville equation. Then from (17) we have

$$f_2(\varphi_1+\varphi_2)=-\mathrm{e}^{\frac{1}{2}(\varphi_1+\varphi_2)}$$
$$f_3(\varphi_1-\varphi_2)=2[\mathrm{e}^{\frac{1}{2}(\varphi_1-\varphi_2)}-\mathrm{e}^{-\frac{1}{2}(\varphi_1-\varphi_2)}]=4\sinh[\tfrac{1}{2}(\varphi_1-\varphi_2)]. \tag{26}$$

Using (18)-(26) yields the formulae of nonlinear superposition

$$\mathrm{e}^{-\frac{1}{2}(\varphi_1+\varphi_2)}\ln\tanh\frac{\varphi_1-\varphi_2}{4}=\gamma_k^2 \qquad k=1,2,\ldots \tag{27}$$

between two solutions of the Liouville equation. Applying (24) to (27) gives the equation of φ_2 as

$$\mathrm{e}^{-\frac{1}{2}\varphi_2}\ln\tanh\frac{2\ln(\sqrt{c}\,R_j^{-1})-\varphi_2}{4}=\sqrt{c}\,R_j^{-1}\gamma_k^2 \qquad j,k=1,2,\ldots. \tag{28}$$

These are some new and complicated solutions to the Liouville equation.

Comparing the above results with the general Bäcklund transformation for the Liouville equation [5, 6], we find that the CBT is a new and simple method for solving the equation.

cbt *of the sinh-Gordon equation.* Let us set $F_3(\varphi_1)=\cosh\varphi_1$, $F_3(\varphi_2)=\cosh\varphi_2$. Then (14) becomes the sinh-Gordon equation and equation (17) implies

$$\begin{aligned} f_4(\varphi_1+\varphi_2) &= \mathrm{i}\,2\sinh\frac{\varphi_1+\varphi_2}{2} \\ f_4(\varphi_1-\varphi_2) &= \mathrm{i}\,2\sinh\frac{\varphi_1-\varphi_2}{2} \end{aligned} \tag{29}$$

since

$$2[F(\varphi_2)-F(\varphi_1)=2(\cosh\varphi_2-\cosh\varphi_1)=-\sinh\frac{\varphi_1+\varphi_2}{2}\sinh\frac{\varphi_1-\varphi_2}{2}.$$

By substituting (29) into (28), one arrives at

$$\ln\tanh\frac{\varphi_1+\varphi_2}{2}\ln\tanh\frac{\varphi_1-\varphi_2}{2}=-\gamma_k^2 \qquad k=1,2,\ldots. \tag{30}$$

The Klein-Gordon equation, Liouville equation and sinh-Gordon equation are very important in physics so that the formulae (21), (27) and (30) of nonlinear superposition for the solutions of these equations will be useful.

We will discuss the canonical Bäcklund transformations of the sine-Gordon, Korteweg-de Vries, nonlinear Schrödinger and some other soliton equations in future publications.

References

[1] Goldstein H 1980 *Classical Mechanics* 2nd edn (New York: Addison-Wesley)
[2] Wenhua H 1989 *J. Phys. A: Math. Gen.* **22** 5033–41
[3] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[4] Bjorken J D and Drell S D 1965 *Relativistic Quantum Fields* (New York: McGraw-Hill)
[5] Leibbrandt G 1980 *Lett. Math. Phys.* **4** 317–21
[6] Leibbrandt G, Wang S and Zamani N 1982 *J. Math. Phys.* **23** 1566–72